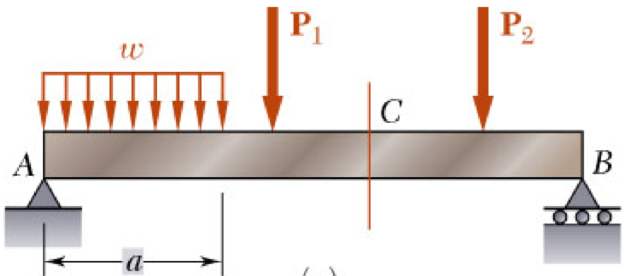
Analysis and Design of Beams for Bending

Introduction



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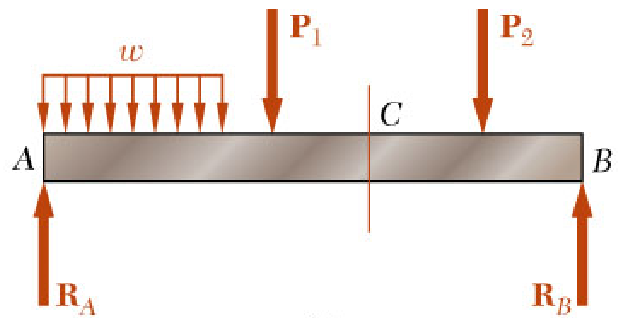

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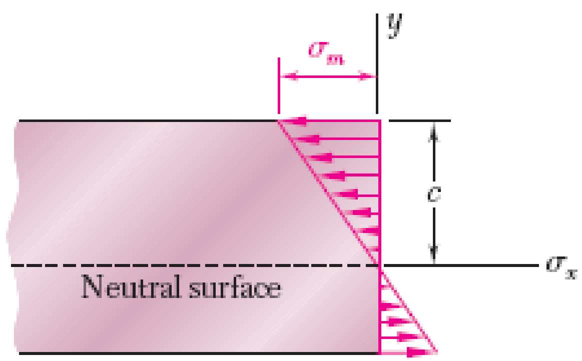


Fig. 4.13

- In this chapter, we will study the analysis and design of beams.
- *Beams* are structural members supporting loads at various points along the member. They are usually long, straight prismatic members.
- In most cases, the loads are perpendicular to the axis of the beam (transverse loading) causing only bending and shear in the beam.
- Normal stress is often the critical design criteria

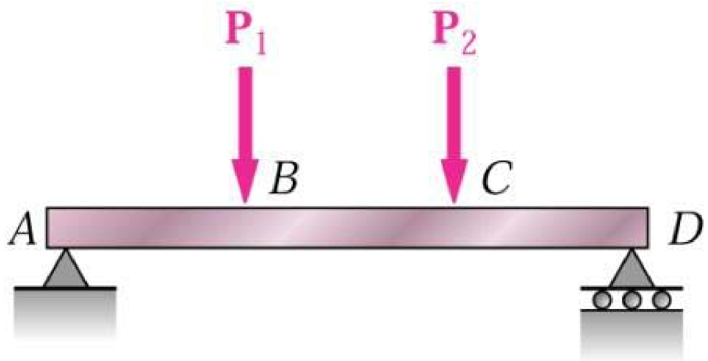
$$\sigma_x = -\frac{My}{I} \qquad \sigma_m = \frac{|M|c}{I} = \frac{|M|}{S}$$

where I is the moment of inertia of the cross section with respect to a centroidal axis perpendicular to the plane of the couple, y is the distance from the neutral surface, and c is the maximum value of that distance (Fig. 4.13). We also recall from Sec. 4.4 that, introducing the elastic section modulus $S = I/c$ of the beam, the maximum value σ_m of the normal stress in the section can be expressed as

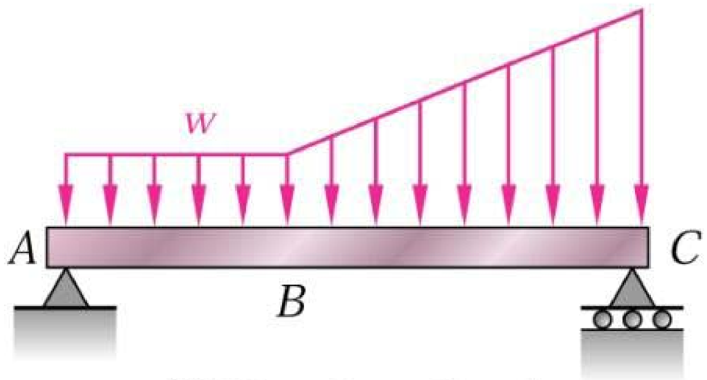
$$\sigma_m = \frac{|M|}{S}$$

Introduction – Classification of Transverse Loading

- Transverse loadings of beams are classified as *concentrated* loads or *distributed* loads.



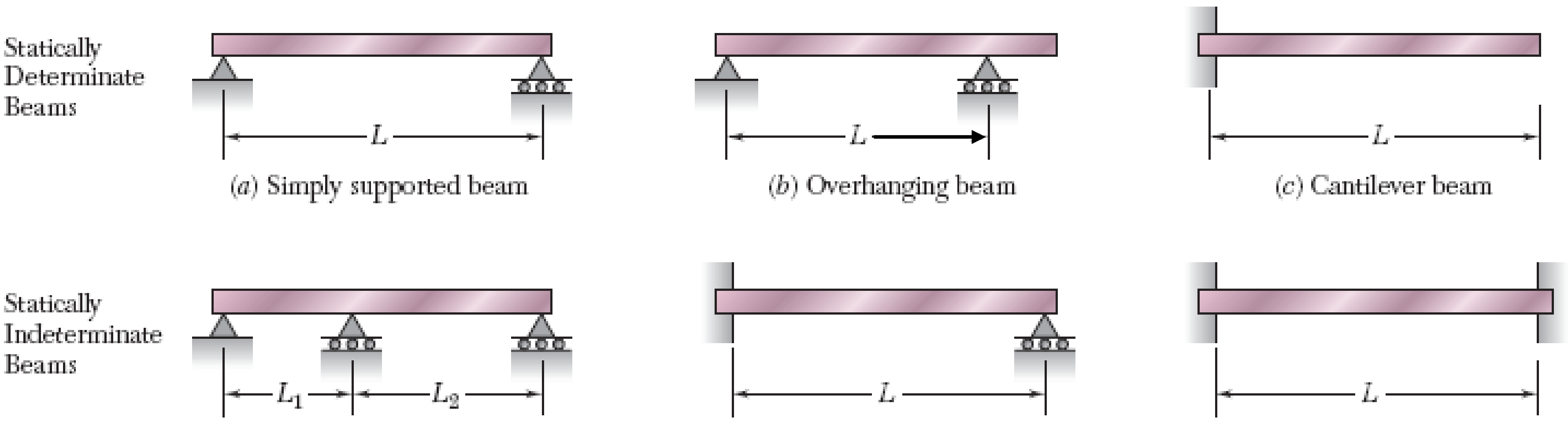
(*a*) Concentrated loads



(*b*) Distributed load

Introduction - Classification of Beams

- Beams are classified according to the way in which they are supported.



(a) Simply supported beam

(b) Overhanging beam

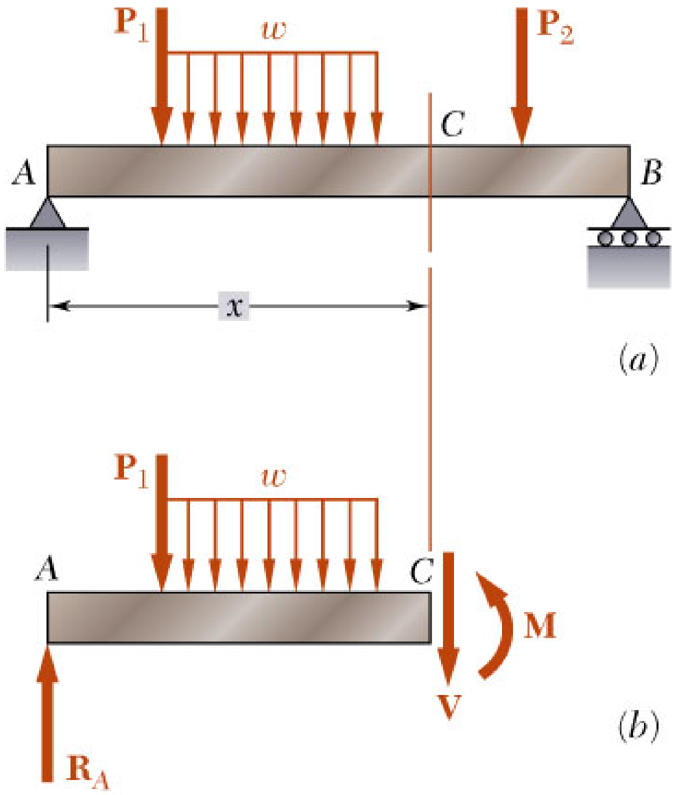
(c) Cantilever beam

(d) Continuous beam

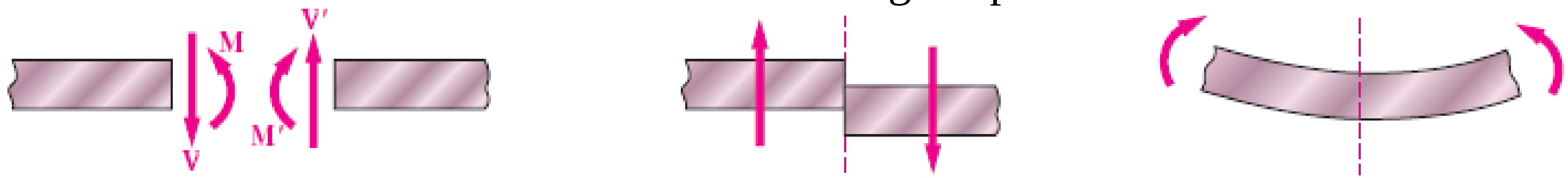
(e) Beam fixed at one end and simply supported at the other end

(f) Fixed beam

Shear and Bending Moment Diagrams



- Determination of maximum normal and shearing stresses requires identification of maximum internal shear force and bending couple.
- Shear force and bending couple at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions on either side of the section.
- Sign conventions for shear forces *V* and *V'* and bending couples *M* and *M'*



(*a*) Internal forces (positive shear and positive bending moment)

(*b*) Effect of external forces (positive shear)

(*c*) Effect of external forces (positive bending moment)

Fig. 5.7

Example 5.01

EXAMPLE 5.01

Draw the shear and bending-moment diagrams for a simply supported beam AB of span L subjected to a single concentrated load **P** at it midpoint C (Fig. 5.8).

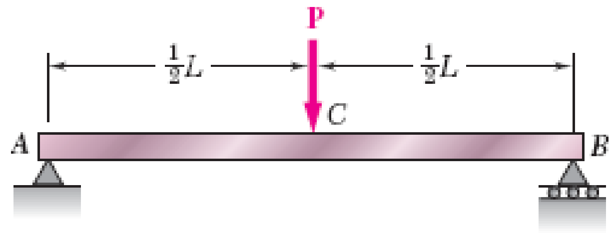
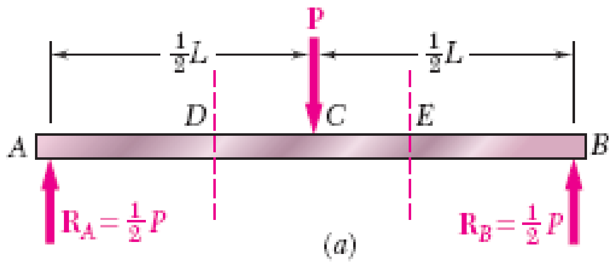


Fig. 5.8

We first determine the reactions at the supports from the free-body diagram of the entire beam (Fig. 5.9a); we find that the magnitude of each reaction is equal to $P/2$.



Example 5.01

Next we cut the beam at a point D between A and C and draw the free-body diagrams of AD and DB (Fig. 5.9b). *Assuming that shear and bending moment are positive,* we direct the internal forces $\mathbf{V}$ and $\mathbf{V}'$ and the internal couples $\mathbf{M}$ and $\mathbf{M}'$ as indicated in Fig. 5.7a. Considering the free body AD and writing that the sum of the vertical components and the sum of the moments about D of the forces acting on the free body are zero, we find $V = +P/2$ and $M = +Px/2$. Both the shear and the bending moment are therefore positive; this may be checked by observing that the reaction at A tends to shear off and to bend the beam at D as indicated in Figs. 5.7b and c. We now plot V and M between A and C (Figs. 5.9d and e); the shear has a constant value $V = P/2$, while the bending moment increases linearly from $M = 0$ at $x = 0$ to $M = PL/4$ at $x = L/2$.

Cutting, now, the beam at a point E between C and B and considering the free body EB (Fig. 5.9c), we write that the sum of the vertical components and the sum of the moments about E of the forces acting on the free body are zero. We obtain $V = -P/2$ and $M = P(L - x)/2$. The shear is therefore negative and the bending moment positive; this can be checked by observing that the reaction at B bends the beam at E as indicated in Fig. 5.7c but tends to shear it off in a manner opposite to that shown in Fig. 5.7b. We can complete, now, the shear and bending-moment diagrams of Figs. 5.9d and e; the shear has a constant value $V = -P/2$ between C and B, while the bending moment decreases linearly from $M = PL/4$ at $x = L/2$ to $M = 0$ at $x = L$.

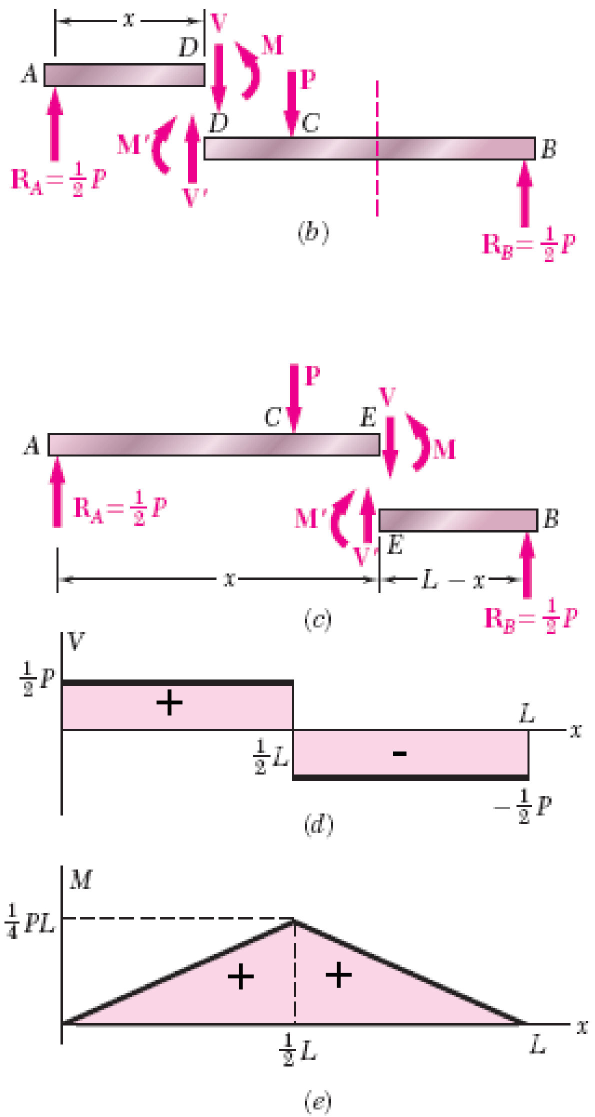


Fig. 5.9

Example 5.02

EXAMPLE 5.02

Draw the shear and bending-moment diagrams for a cantilever beam AB of span L supporting a uniformly distributed load w (Fig. 5.10).

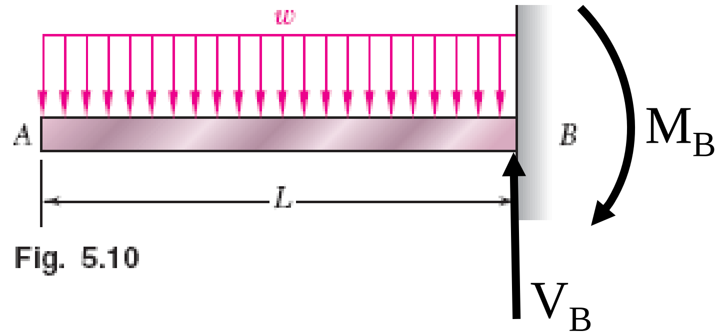
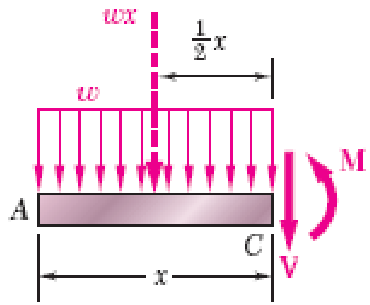


Fig. 5.10

We cut the beam at a point C between A and B and draw the free-body diagram of AC (Fig. 5.11a), directing **V** and **M** as indicated in Fig. 5.7a. Denoting by x the distance from A to C and replacing the distributed load over AC by its resultant wx applied at the midpoint of AC, we write



(a)

Example 5.02

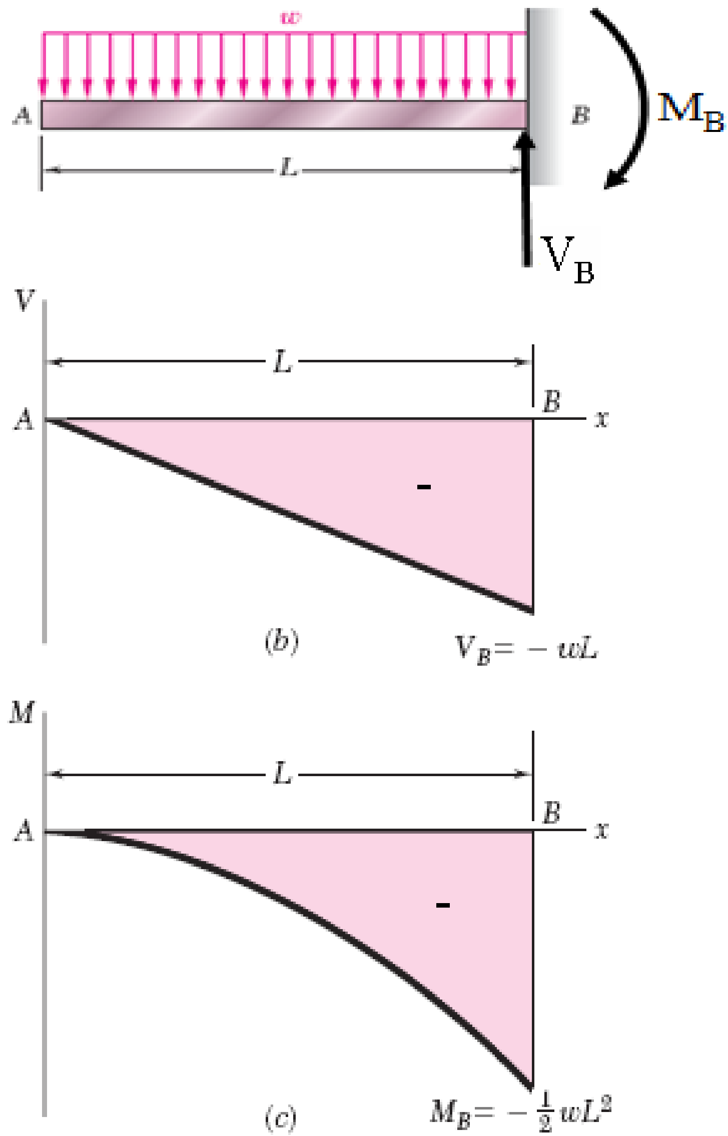


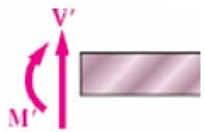
Fig. 5.11

$$+\uparrow \Sigma F_y = 0: \qquad -wx - V = 0 \qquad V = -wx$$

$$+\circlearrowleft \Sigma M_C = 0: \qquad wx\left(\frac{x}{2}\right) + M = 0 \qquad M = -\frac{1}{2}wx^2$$

We note that the shear diagram is represented by an oblique straight line (Fig. 5.11*b*) and the bending-moment diagram by a parabola (Fig. 5.11*c*). The maximum values of V and M both occur at B, where we have

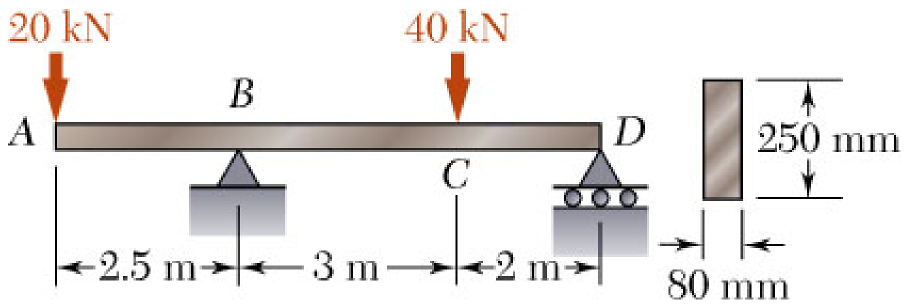
$$V_B = -wL \qquad M_B = -\tfrac{1}{2}wL^2$$



While drawing the V and M diagrams from left to right

If we have a clockwise external moment, we will go up in M-diagram.

Sample Problem 5.1

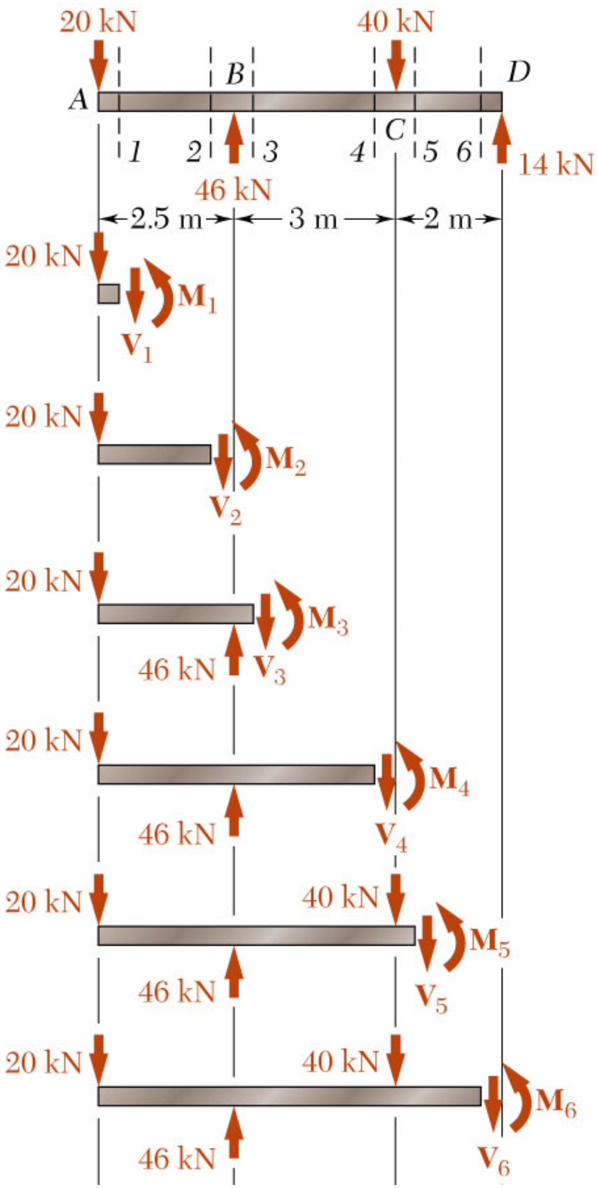


For the timber beam and loading shown, draw the shear and bend-moment diagrams and determine the maximum normal stress due to bending.

SOLUTION:

- From the FBD of entire beam, determine the reaction forces.
- Section the beam at points near supports and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples.
- Identify the maximum shear and bending-moment from plots of their distributions.
- Apply the elastic flexure formula to determine the corresponding maximum normal stress.

Sample Problem 5.1



SOLUTION:

- From the FBD of entire beam, determine the reaction forces as below;

$$\sum F_y = 0 \quad \& \quad \sum M_B = 0: \quad R_B = 46\,\text{kN} \quad R_D = 14\,\text{kN}$$

- Section the beam and apply equilibrium analyses on resulting free-bodies

$$\sum F_y = 0 \quad -20\text{ kN} - V_1 = 0 \qquad V_1 = -20\,\text{kN}$$

$$\sum M_1 = 0 \quad (20\,\text{kN})(0\,\text{m}) + M_1 = 0 \quad M_1 = 0$$

$$\sum F_y = 0 \quad -20\text{ kN} - V_2 = 0 \qquad V_2 = -20\,\text{kN}$$

$$\sum M_2 = 0 \quad (20\,\text{kN})(2.5\,\text{m}) + M_2 = 0 \quad M_2 = -50\,\text{kN}\cdot\text{m}$$

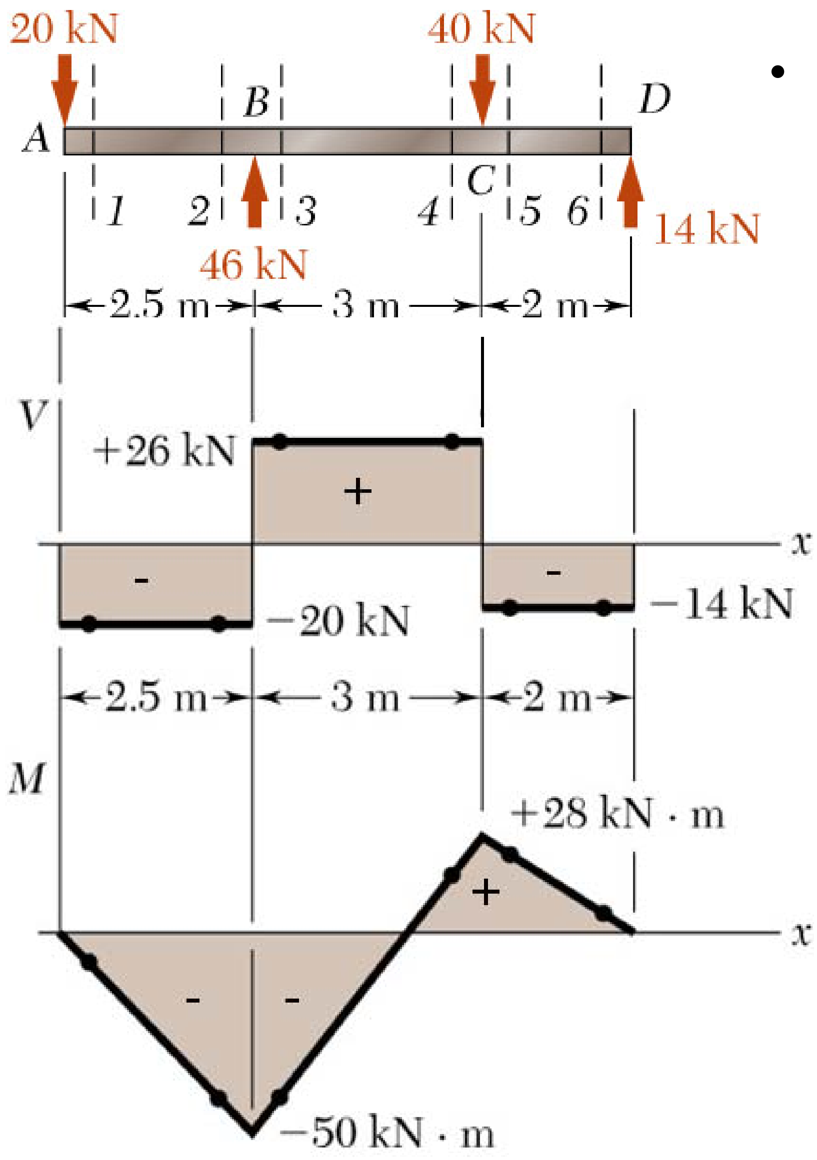
$$V_3 = +26\,\text{kN} \quad M_3 = -50\,\text{kN}\cdot\text{m}$$

$$V_4 = +26\,\text{kN} \quad M_4 = +28\,\text{kN}\cdot\text{m}$$

$$V_5 = -14\,\text{kN} \quad M_5 = +28\,\text{kN}\cdot\text{m}$$

$$V_6 = -14\,\text{kN} \quad M_6 = 0$$

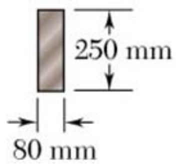
Sample Problem 5.1



- Identify the maximum shear and bending-moment from plots of their distributions.

$$V_m = 26\,\text{kN} \quad M_m = |M_B| = 50\,\text{kN}\cdot\text{m}$$

- Apply the elastic flexure formula to determine the corresponding maximum normal stress.



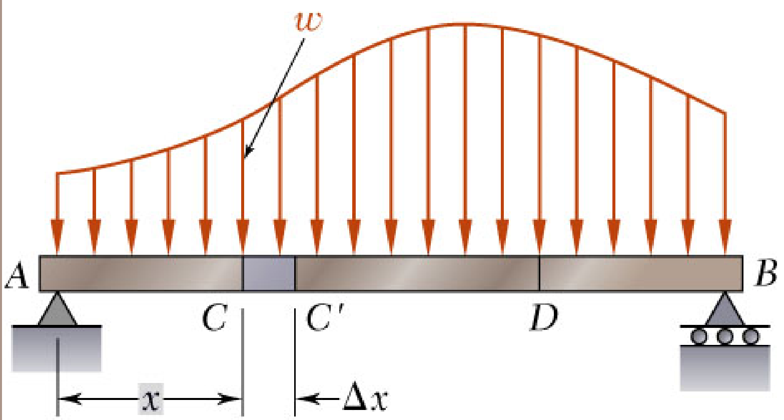
$$S = \tfrac{1}{6}bh^2 = \tfrac{1}{6}(0.080\,\text{m})(0.250\,\text{m})^2$$

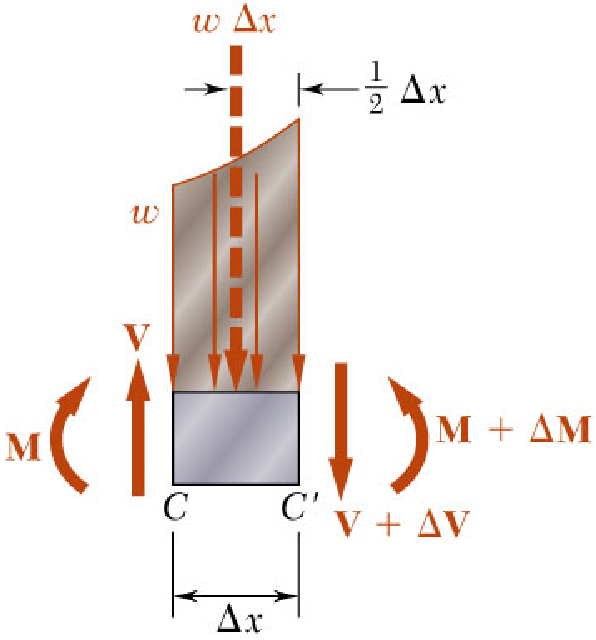
$$= 833.33\times10^{-6}\,\text{m}^3$$

$$\sigma_m = \frac{|M_B|}{S} = \frac{50\times10^3\,\text{N}\cdot\text{m}}{833.33\times10^{-6}\,\text{m}^3}$$

$$\sigma_m = 60.0\times10^6\,\text{Pa}$$ = 60 MPa

Relations Among Load, Shear and Bending Moment





- Relationship between load and shear:

$$\sum F_y = 0: \quad V - (V + \Delta V) - w\,\Delta x = 0$$

$$\Delta V = -w\,\Delta x$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\frac{dV}{dx} = -w \qquad (5.5)$$

Integrating (5.5) between points C and D, we write

$$V_D - V_C = -\int_{x_C}^{x_D} w\,dx \qquad (5.6)$$

$$V_D - V_C = -(\text{area under load curve between } C \text{ and } D) \qquad (5.6')$$

- Relationship between shear and bending moment:

$$\sum M_{C'} = 0: \quad (M + \Delta M) - M - V\,\Delta x + w\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \tfrac{1}{2} w (\Delta x)^2$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\frac{dM}{dx} = V \qquad (5.7)$$

Integrating (5.7) between points C and D, we write

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx \qquad (5.8)$$

$$M_D - M_C = \text{area under shear curve between } C \text{ and } D \qquad (5.8')$$

Example 5.03

EXAMPLE 5.03

Draw the shear and bending-moment diagrams for the simply supported beam shown in Fig. 5.13 and determine the maximum value of the bending moment.

From the free-body diagram of the entire beam, we determine the magnitude of the reactions at the supports.

$$R_A = R_B = \tfrac{1}{2}wL$$

Next, we draw the shear diagram. Close to the end A of the beam, the shear is equal to R_A, that is, to $\frac{1}{2}wL$, as we can check by considering as a free body a very small portion of the beam.

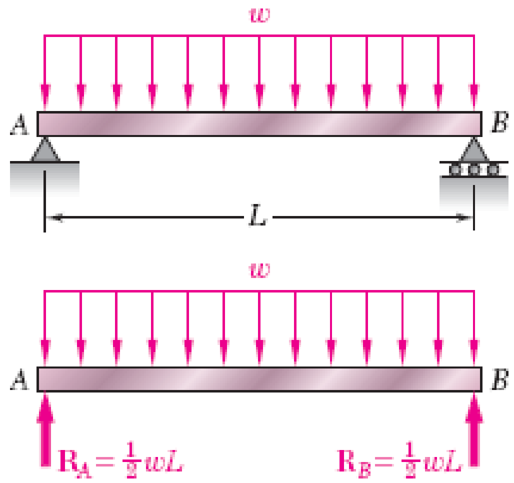
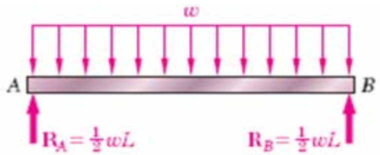


Fig. 5.13

End

Example 5.03



Using Eq. (5.6), we then determine the shear V at any distance x from A; we write

$$V - V_A = -\int_0^x w\,dx = -wx$$

$$V = V_A - wx = \tfrac{1}{2}wL - wx = w(\tfrac{1}{2}L - x)$$

The shear curve is thus an oblique straight line which crosses the x axis at $x = L/2$ (Fig. 5.14a). Considering, now, the bending moment, we first observe that $M_A = 0$. The value M of the bending moment at any distance x from A may then be obtained from Eq. (5.8); we have

$$M - M_A = \int_0^x V\,dx$$

$$M = \int_0^x w(\tfrac{1}{2}L - x)\,dx = \tfrac{1}{2}w(Lx - x^2)$$

The bending-moment curve is a parabola. The maximum value of the bending moment occurs when $x = L/2$, since V (and thus dM/dx) is zero for that value of x. Substituting $x = L/2$ in the last equation, we obtain $M_{max} = wL^2/8$ (Fig. 5.14b).

$V = -w.x+c_1$

$M = -w.x^2/2+c_1.x+c_2$

At x=0, M=0 Then $c_2=0$

At x=L, M=0 Then $c_1=w.L/2$

Finally; $V = -w.x + w.L/2 = w.[(L/2)-x]$ and

$M = -w.x^2/2+w.x.L/2 = (w/2).(L.x-x^2)$

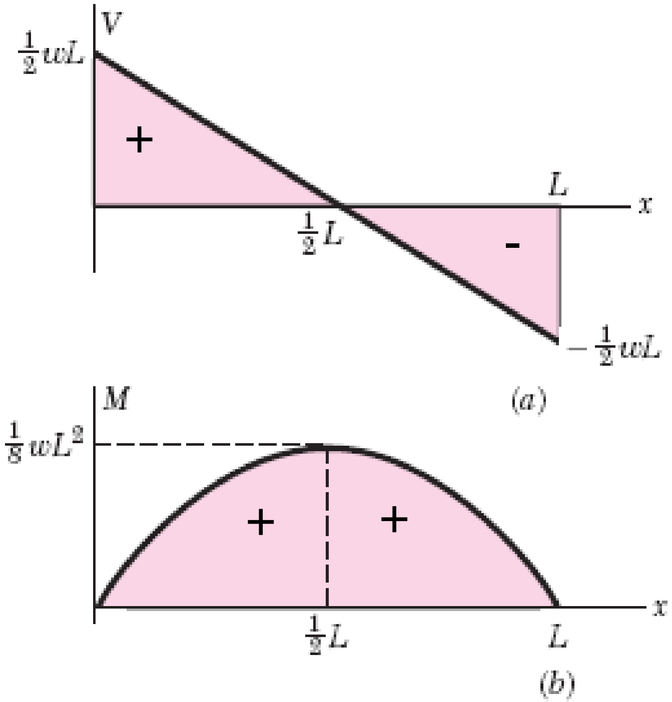
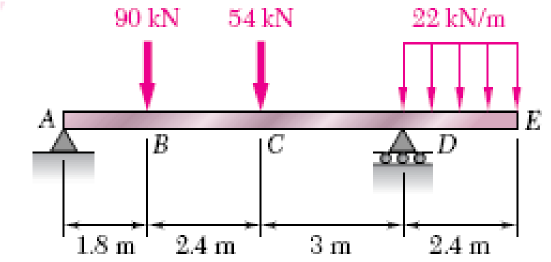


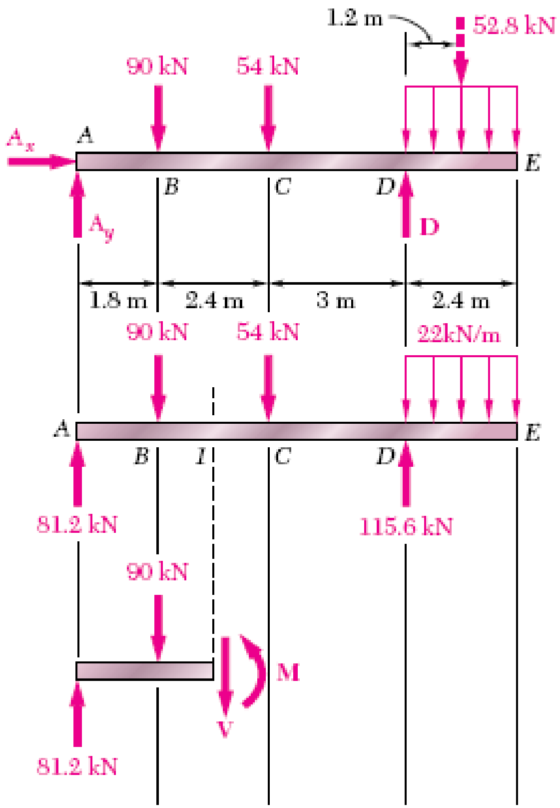
Fig. 5.14

Sample Problem 5.3



SAMPLE PROBLEM 5.3

Draw the shear and bending-moment diagrams for the beam and loading shown.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$+\circlearrowleft \Sigma M_A = 0$:

D(7.2 m) − (90 kN)(1.8 m) − (54 kN)(4.2 m) − (52.8 kN)(8.4 m) = 0

$D = 115.6$ kN **D** = 115.6 kN ↑

$+\uparrow \Sigma F_y = 0$: $A_y - 90$ kN − 54 kN + 115.6 kN − 52.8 kN = 0

$A_y = +\ 81.2$ kN $\mathbf{A_y} = 081.2$ kN ↑

$\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

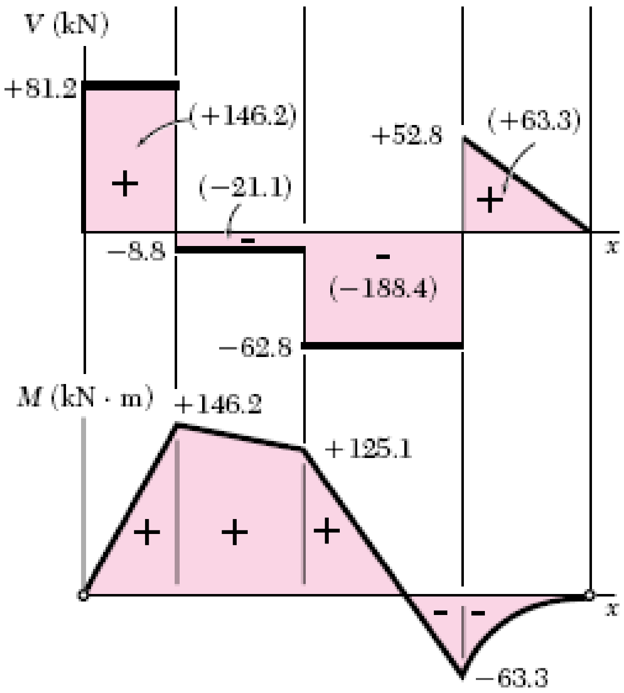
We also note that at both A and E the bending moment is zero; thus, two points (indicated by dots) are obtained on the bending-moment diagram.

Shear Diagram. Since $dV/dx = -w$, we find that between concentrated loads and reactions the slope of the shear diagram is zero (i.e., the shear is constant). The shear at any point is determined by dividing the beam into two parts and considering either part as a free body. For example, using the portion of beam to the left of section *1*, we obtain the shear between B and C:

$+\uparrow \Sigma F_y = 0$: +81.2 kN − 90 kN − V = 0 $V = -8.8$ kN

We also find that the shear is +52.8 kN just to the right of D and zero at end E. Since the slope $dV/dx = -w$ is constant between D and E, the shear diagram between these two points is a straight line.

Sample Problem 5.3



Bending-Moment Diagram. We recall that the area under the shear curve between two points is equal to the change in bending moment between the same two points. For convenience, the area of each portion of the shear diagram is computed and is indicated in parentheses on the diagram. Since the bending moment M_A at the left end is known to be zero, we write

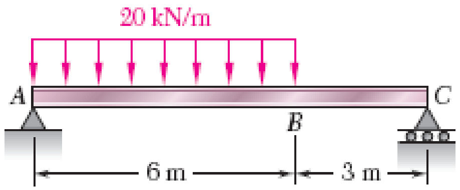
$$M_B - M_A = +146.2 \qquad M_B = +146.2 \text{ kN} \cdot \text{m}$$
$$M_C - M_B = -21.1 \qquad M_C = +125.1 \text{ kN} \cdot \text{m}$$
$$M_D - M_C = -188.4 \qquad M_D = -63.3 \text{ kN} \cdot \text{m}$$
$$M_E - M_D = +63.3 \qquad M_E = 0$$

Since M_E is known to be zero, a check of the computations is obtained.

Between the concentrated loads and reactions the shear is constant; thus, the slope dM/dx is constant and the bending-moment diagram is drawn by connecting the known points with straight lines. Between D and E where the shear diagram is an oblique straight line, the bending-moment diagram is a parabola.

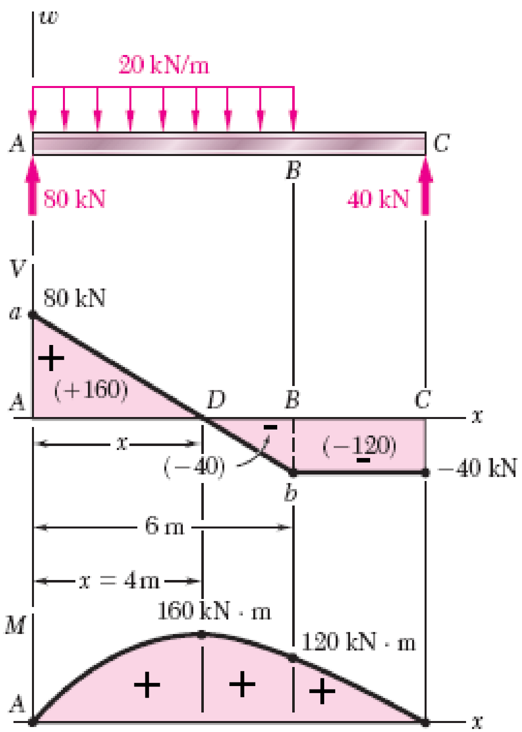
From the V and M diagrams we note that $V_{max} = 81.2$ kN and $M_{max} = 146.2$ kN · m.

Sample Problem 5.4



SAMPLE PROBLEM 5.4

The W360 × 79 rolled-steel beam AC is simply supported and carries the uniformly distributed load shown. Draw the shear and bending-moment diagrams for the beam and determine the location and magnitude of the maximum normal stress due to bending.



Alternatively, x can be found by using similar triangles.

SOLUTION

Reactions. Considering the entire beam as a free body, we find

$$\mathbf{R}_A = 80 \text{ kN} \uparrow \qquad \mathbf{R}_C = 40 \text{ kN} \uparrow$$

Shear Diagram. The shear just to the right of A is $V_A = +80$ kN. Since the change in shear between two points is equal to *minus* the area under the load curve between the same two points, we obtain V_B by writing

$$V_B - V_A = -(20 \text{ kN/m})(6 \text{ m}) = -120 \text{ kN}$$
$$V_B = -120 + V_A = -120 + 80 = -40 \text{ kN}$$

The slope $dV/dx = -w$ being constant between A and B, the shear diagram between these two points is represented by a straight line. Between B and C, the area under the load curve is zero; therefore,

$$V_C - V_B = 0 \qquad V_C = V_B = -40 \text{ kN}$$

and the shear is constant between B and C.

Bending-Moment Diagram. We note that the bending moment at each end of the beam is zero. In order to determine the maximum bending moment, we locate the section D of the beam where $V = 0$. We write

$$V_D - V_A = -wx$$
$$0 - 80 \text{ kN} = -(20 \text{ kN/m})x$$

and, solving for x: $x = 4$ m ◀

Sample Problem 5.4

The maximum bending moment occurs at point D, where we have $dM/dx = V = 0$. The areas of the various portions of the shear diagram are computed and are given (in parentheses) on the diagram. Since the area of the shear diagram between two points is equal to the change in bending moment between the same two points, we write

$$
\begin{aligned}
M_D - M_A &= +160\,\mathrm{kN\cdot m} & M_D &= +160\,\mathrm{kN\cdot m}\\
M_B - M_D &= -\ \ 40\,\mathrm{kN\cdot m} & M_B &= +120\,\mathrm{kN\cdot m}\\
M_C - M_B &= -120\,\mathrm{kN\cdot m} & M_C &= 0
\end{aligned}
$$

The bending-moment diagram consists of an arc of parabola followed by a segment of straight line; the slope of the parabola at A is equal to the value of V at that point.

Maximum Normal Stress. It occurs at D, where $|M|$ is largest. From Appendix C we find that for a W360 × 79 rolled-steel shape, S = **1280x10³ mm³** about a horizontal axis. Substituting this value and $|M| = |M_D| = 160 \times 10^3\ \mathrm{N\cdot m}$ into Eq. (5.3), we write

$$\sigma_m = \frac{|M_D|}{S} = \frac{160 \times 10^3\ \mathrm{N\cdot m}}{1280 \times 10^{-6}\ \mathrm{m}^3} = 125.0 \times 10^6\ \mathrm{Pa}$$

Maximum normal stress in the beam = 125.0 MPa ◄

Design of Prismatic Beams for Bending

- The largest normal stress is found at the surface where the maximum bending moment occurs.

$$\sigma_m = \frac{|M|_{\max} c}{I} = \frac{|M|_{\max}}{S}$$

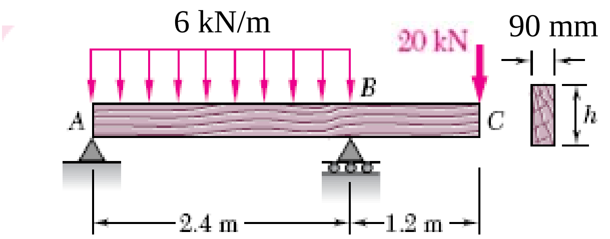
- A safe design requires that the maximum normal stress be less than the allowable stress for the material used. This criteria leads to the determination of the minimum acceptable section modulus.

$$\sigma_m \le \sigma_{all}$$

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}}$$

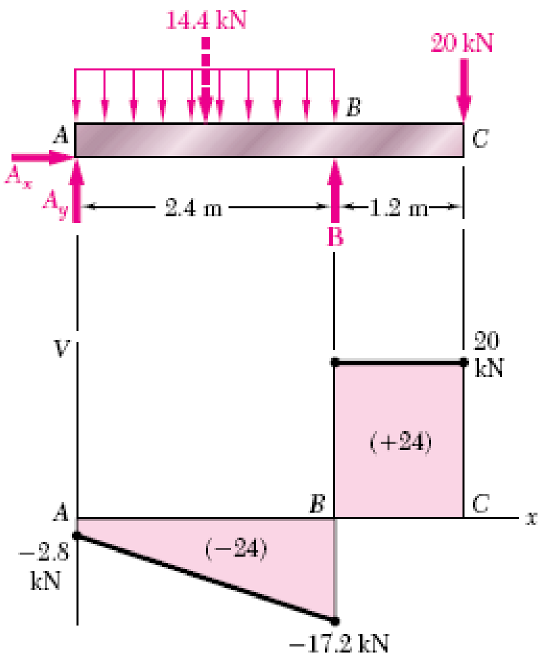
- Among beam section choices which have an acceptable section modulus, the one with the smallest weight per unit length or cross sectional area will be the least expensive (i.e. the cheapest) and the best choice for the design.

Sample Problem 5.7



SAMPLE PROBLEM 5.7

A 3.6 m-long overhanging timber beam AC with an 2.4-m span AB is to be designed to support the distributed and concentrated loads shown. Knowing that timber of 100-mm nominal width (90-mm actual width) with a 12-MPa allowable stress is to be used, determine the minimum required depth h of the beam.



SOLUTION

Reactions. Considering the entire beam as a free body, we write

$$+\circlearrowleft \Sigma M_A = 0:\ B(2.4\text{ m}) - (14.4\text{ kN})(1.2\text{ m}) - (20\text{ kN})(3.6\text{ m}) = 0$$

$$B = 37.2\text{ kN} \qquad \mathbf{B} = 37.2\text{ kN}\uparrow$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0: \qquad A_x = 0$$

$$+\uparrow \Sigma F_y = 0:\ A_y + 37.2\text{ kN} - 14.4\text{ kN} - 20\text{ kN} = 0$$

$$A_y = -2.8\text{ kN} \qquad \mathbf{A} = 2.8\text{ kN}\downarrow$$

Shear Diagram. The shear just to the right of A is $V_A = A_y = -2.8$ kN. Since the change in shear between A and B is equal to *minus* the area under the load curve between these two points, we obtain V_B by writing

$$V_B - V_A = -(6\text{ kN/m})(2.4\text{ m}) = -14.4\text{ kN}$$

$$V_B = V_A - 14.4\text{ kN} = -2.8\text{ kN} - 14.4\text{ kN} = -17.2\text{ kN}$$

The reaction at B produces a sudden increase of 37.2 kN in V, resulting in a value of the shear equal to 20 kN to the right of B. Since no load is applied between B and C, the shear remains constant between these two points.

Sample Problem 5.7

Determination of $|M|_{max}$. We first observe that the bending moment is equal to zero at both ends of the beam: $M_A = M_C = 0$. Between A and B the bending moment decreases by an amount equal to the area under the shear curve, and between B and C it increases by a corresponding amount. Thus, the maximum absolute value of the bending moment is $|M|_{max} = 24\ \text{kN} \cdot \text{m}$.

Minimum Allowable Section Modulus. Substituting into Eq. (5.9) the given value of σ_{all} and the value of $|M|_{max}$ that we have found, we write

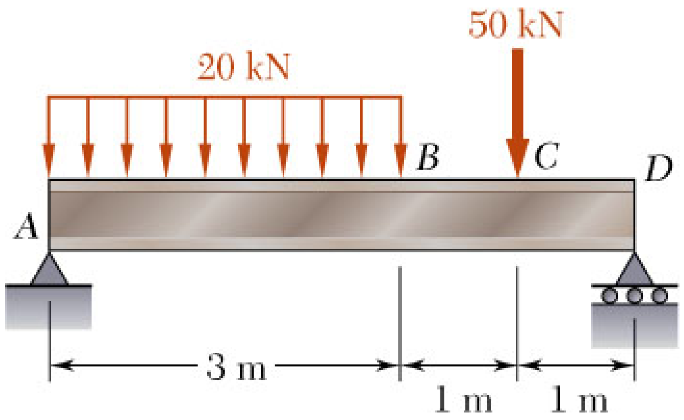
$$S_{min} = \frac{|M|_{max}}{\sigma_{all}} = \frac{24\ \text{kN} \cdot \text{m}}{12\ \text{MPa}} = 2 \times 10^6\ \text{mm}^3$$

Minimum Required Depth of Beam. Recalling the formula developed in part 4 of the design procedure described in Sec. 5.4 and substituting the values of b and S_{min}, we have

$$\tfrac{1}{6} bh^2 \geq S_{min} \qquad \tfrac{1}{6}(90\ \text{mm})h^2 \geq 2 \times 10^6\ \text{mm}^3 \qquad h \geq 365.2\ \text{mm}$$

The minimum required depth of the beam is $h = 366\ \text{mm}$ ◀

Sample Problem 5.8

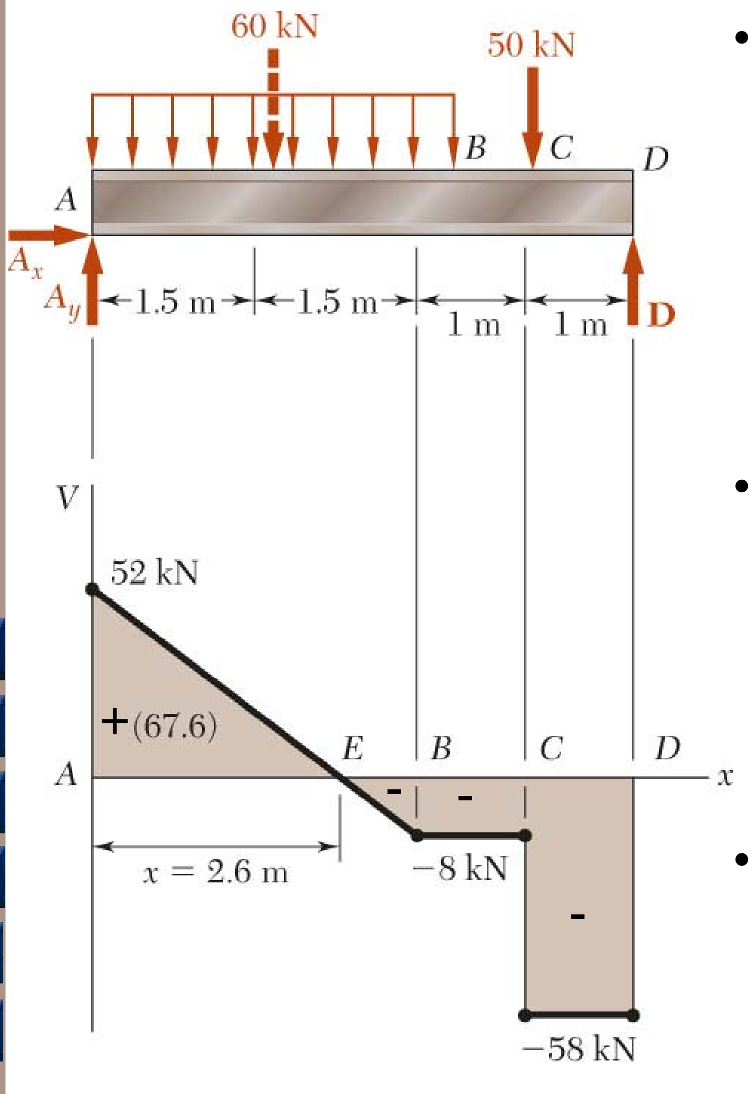


A simply supported steel beam is to carry the distributed and concentrated loads shown. Knowing that the allowable normal stress for the grade of steel to be used is 160 MPa, select the wide-flange shape that should be used.

SOLUTION:

- Considering the entire beam as a free-body, determine the reactions at *A* and *D*.
- Develop the shear diagram for the beam and load distribution. From the diagram, determine the maximum bending moment.
- Determine the minimum acceptable beam section modulus. Choose the best standard section which meets this criteria.

Sample Problem 5.8



- Considering the entire beam as a free-body, determine the reactions at *A* and *D*.

$$\sum M_A = 0 = D(5\,\text{m}) - (60\,\text{kN})(1.5\,\text{m}) - (50\,\text{kN})(4\,\text{m})$$
$$D = 58.0\,\text{kN}$$
$$\sum F_y = 0 = A_y + 58.0\,\text{kN} - 60\,\text{kN} - 50\,\text{kN}$$
$$A_y = 52.0\,\text{kN}$$

- Develop the shear diagram and determine the maximum bending moment.

$$V_A = A_y = 52.0\,\text{kN}$$
$$V_B - V_A = -(\text{area under load curve}) = -60\,\text{kN}$$
$$V_B = -8\,\text{kN}$$

- Maximum bending moment occurs at $V = 0$ or $x = 2.6$ m.

$$|M|_{max} = (\text{area under shear curve}, A \text{ to } E)$$
$$= 67.6\,\text{kN}$$